國立臺北教育大學 102 學年度學士班轉學考試

學系（組）：數學暨資訊教育學系（數學組）

年　　級：大三

科　　目：高等微積分

1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. If there is an $a \in (0,1)$ satisfying

$$|x_{n+1} - x_n| \le a^n$$

For all $n \in \mathbb{N}$, show that $x_n \to \alpha$ as $n \to \infty$ for some $\alpha \in \mathbb{R}$. (15%)

2. Define

$$f_\alpha(x) = \begin{cases} |x|^\alpha \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that $f_\alpha(x)$ is continuous at $x = 0$ when $\alpha > 0$ and is differentiable at $x = 0$ when $\alpha > 1$. (15%)

3. For each $n \in \mathbb{N}$, define

$$a_n \equiv \left(\frac{(2n)!}{n!\,n^n}\right)^{1/n}.$$

Show that $a_n \to 4/e$ as $n \to \infty$. (15%)

4. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences in $\mathbb{R}$ and satisfy

$$\sum_{n=1}^{\infty} a_n^2 < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} b_n^2 < \infty.$$

Show that the infinite series $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely. (15%)

5. Let E be a nonempty subset of $\mathbb{R}^n$. Show that E is closed if and only if E contains all its limit points. (15%)

6. Prove that

$$f(x,y) = \begin{cases} \dfrac{x^3 - xy^2}{x^2 + y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0), \end{cases}$$

is continuous on $\mathbb{R}^2$ and has first-order partial derivatives everywhere on $\mathbb{R}^2$, but f is not differentiable at $(0,0)$. (15%)

7. Show that if $f : [a,b] \to [a,b]$ is continuous, then f has a fixed point in $[a,b]$. (10%)